Final Exam — Functional Analysis (WBMA033-05)

Wednesday 28 June 2023, 8.30h–10.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 points)

Recall the following linear space from the lecture notes:

$$\ell^1 = \left\{ x = (x_1, x_2, x_3, \dots) \, : \, x_k \in \mathbb{K}, \, \sum_{k=1}^{\infty} |x_k| < \infty \right\}.$$

We can equip this space with the following norms:

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k| \quad \text{and} \quad \|x\|_w = \sum_{k=1}^{\infty} e^{\sin(k)} |x_k|.$$

Show that these norms are equivalent.

Problem 2 (10 + 5 + 10 + 5 = 30 points)

Consider the following linear operator:

$$T : \mathcal{C}([0,1], \mathbb{K}) \to \mathcal{C}([0,1], \mathbb{K}), \quad Tf(x) = f(x^2).$$

On the space $\mathcal{C}([0,1], \mathbb{K})$ we take the sup-norm $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$.

(a) Compute the operator norm of T.

(b) Show that $\lambda = 1$ is an eigenvalue of T.

(c) Show that T is invertible.

(d) Is T compact?

Problem 3 (10 + 10 = 20 points)

Let X be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let $T \in B(X)$ be of the form $Tx = \langle x, u \rangle v$, where $u, v \in X$ are nonzero.

(a) Show that $T^* y = \langle y, v \rangle u$ for all $y \in X$.

(b) Assume that $u = cv$ for some $c \in \mathbb{C}$. Show that T is selfadjoint if and only if $c \in \mathbb{R}$.

Turn page for problems 4 and 5!

Problem 4 (10 + 5 = 15 points)

(a) Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. Prove that the following statements are equivalent:

(i) T is bounded;

(ii) if (x_n) is a sequence in X such that $x_n \to 0$ and $Tx_n \to y$, then $y = 0$.

Hint: use the Closed Graph Theorem.

(b) Now assume that X is a Hilbert space over $\mathbb{C}$ and that the linear operator $T : X \to X$ satisfies the following property:

$$|\langle Tx, z\rangle| \leq \|x\| \, \|z\| \quad \text{for all} \quad x, z \in X.$$

Use part (a) to prove that T is bounded.

Problem 5 (15 points)

Equip the linear space $X = \mathcal{C}([-1, 1], \mathbb{C})$ with the following norm:

$$\|f\| = \int_{-1}^{1} |f(x)| \, dx, \qquad f \in X.$$

Let $g(x) = e^{-5ix}$. Prove that there exists a functional $\varphi \in X'$ such that

$$\varphi(g) = 6 + 4i \qquad \text{and} \qquad \|\varphi\| = \sqrt{13}.$$

End of test (90 points)

Solution of problem 1 (10 points)

Let $x \in \ell^1$ be arbitrary. Note that for all $k \in \mathbb{N}$ we have $-1 \le \sin(k) \le 1$ and therefore

$$e^{-1}|x_k| \le e^{\sin(k)}|x_k| \le e|x_k|.$$

(5 points)

Summing over all $k \in \mathbb{N}$ gives

$$e^{-1}\sum_{k=1}^{\infty}|x_k| \le \sum_{k=1}^{\infty} e^{\sin(k)}|x_k| \le e\sum_{k=1}^{\infty}|x_k|,$$

which shows that $e^{-1}\|x\|_1 \le \|x\|_w \le e\|x\|_1$. Since $x \in \ell^1$ is arbitrary, this precisely means that the two norms are equivalent.
(5 points)

Solution of problem 2 (10 + 5 + 10 + 5 = 30 points)

(a) Since the function $x \mapsto x^2$ maps the interval $[0,1]$ bijectively onto itself we have

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |Tf(x)| = \sup_{x\in[0,1]} |f(x^2)| = \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty.$$

(7 points)

Therefore, the operator norm of T is given by

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} = 1.$$

(3 points)

(b) The equality $f(x) = f(x^2)$ holds for all constant functions. Therefore, any nonzero constant function f is an eigenvector for the eigenvalue $\lambda = 1$.
(5 points)

(c) Consider the operator

$$S : \mathcal{C}([0,1],\mathbb{K}) \to \mathcal{C}([0,1],\mathbb{K}), \quad Sf(x) = f(\sqrt{x}).$$

We have

$$STf(x) = f(\sqrt{x^2}) = f(x) \quad \text{and} \quad TSf(x) = f(\sqrt{x}^2) = f(x),$$

which means that $ST = TS = I$.
(7 points)

By a similar argument as in part (a) it follows that S is bounded. Therefore, the operator T is invertible.
(3 points)

(d) *Method 1.* The space $\mathcal{C}([0,1],\mathbb{K})$ is infinite-dimensional. If T were compact, then we would have $0 \in \sigma(T)$. However, in part (c) we have established that T is invertible, which means that $0 \in \rho(T)$. Therefore, T is not compact.
(5 points)

Method 2. If T is compact, then so is $I = TT^{-1}$. But then the closed unit ball is compact. However, this is not possible because the space $\mathcal{C}([0,1],\mathbb{K})$ is infinite-dimensional. Therefore, T is not compact.
(5 points)

Solution of problem 3 (10 + 10 = 20 points)

Let X be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let $T \in B(X)$ be of the form $Tx = \langle x, u \rangle v$, where $u, v \in X$ are nonzero.

(a) Show that $T^*y = \langle y, v \rangle u$ for all $y \in X$.

For all $x, y \in X$ we have

$$\begin{aligned} \langle Tx, y \rangle &= \langle \langle x, u \rangle v, y \rangle \\ &= \langle x, u \rangle \langle v, y \rangle \\ &= \langle x, u \rangle \overline{\langle y, v \rangle} \\ &= \langle x, \langle y, v \rangle u \rangle, \end{aligned}$$

which shows that $T^*y = \langle y, v \rangle u$.
(10 points; 2 points per correct equality; 2 points for conclusion)

(b) Assuming that $u = cv$ for some $c \in \mathbb{C}$ gives

$$Tx = \bar{c}\langle x, v \rangle v \quad \text{and} \quad T^*x = c\langle x, v \rangle v,$$

for all $x \in X$. If $c \in \mathbb{R}$, then $\bar{c} = c$ so that $Tx = T^*x$ for all $x \in X$ which shows that T is selfadjoint.
(5 points)

Conversely, if T is selfadjoint, then $Tx = T^*x$ for all $x \in X$. In particular, we have $Tv = T^*v$, or, equivalently,

$$\bar{c}\|v\|^2 v = c\|v\|^2 v,$$

which implies that $\bar{c} = c$ and thus $c \in \mathbb{R}$.
(5 points)

Solution of problem 4 (10 + 5 = 15 points)

(a) *Proof of (i) ⇒ (ii).* Assume that T is bounded. Let (x_n) be a sequence such that $x_n \to 0$ and $Tx_n \to y$. Then it follows that

$$\|y\| = \|y - Tx_n + Tx_n\| \leq \|y - Tx_n\| + \|Tx_n\| \leq \|y - Tx_n\| + \|T\| \, \|x_n\|.$$

Since the right-hand side tends to zero, it follows that $y = 0$.
(5 points)

Proof of (ii) ⇒ (i). Assume that $x_n \to x$ and $Tx_n \to y$. Introduce the new sequence $z_n = x_n - x$. Then it follows that $z_n \to 0$ and $Tz_n \to y - Tx$. By assumption it follows that $y - Tx = 0$ so that $y = Tx$. We conclude that the graph of T is closed. Since X and Y are Banach spaces we can apply the Closed Graph Theorem with $V = X$ to conclude that T is bounded.
(5 points)

(b) Let $z \in X$ be arbitrary, and let (x_n) be a sequence in X such that $x_n \to 0$ and $Tx_n \to y$. On the one hand, we have that

$$|\langle Tx_n, z\rangle| \leq \|x_n\| \, \|z\| \to 0.$$

On the other hand, we have that

$$\langle Tx_n, z\rangle \to \langle y, z\rangle.$$

(3 points)

By uniqueness of limits, we conclude that $\langle y, z\rangle = 0$. Since $z \in X$ was arbitrary, it follows that $y \in X^\perp = \{0\}$ so that $y = 0$. By part (a) we conclude that T is bounded.
(2 points)

Solution of problem 5 (15 points)

Define the map

$$\varphi : \operatorname{span}\{g\} \to \mathbb{C}, \quad \varphi(\lambda g) = \lambda(6+4i).$$

With $\lambda = 1$ we have that $\varphi(g) = 6 + 4i$.
(2 points)

Since $\|g\| = 2$ we have that

$$\|\varphi\| = \sup_{\lambda \neq 0} \frac{|\varphi(\lambda g)|}{\|\lambda g\|} = \sup_{\lambda \neq 0} \frac{|\lambda|\sqrt{52}}{2|\lambda|} = \sqrt{13}.$$

(8 points)

Now apply the Hahn-Banach theorem to extend φ to the entire space X while preserving the norm.
(5 points)